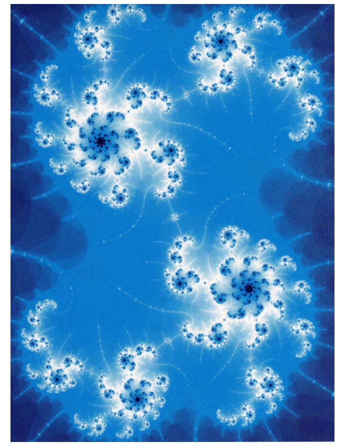
3.12H Complex Computations

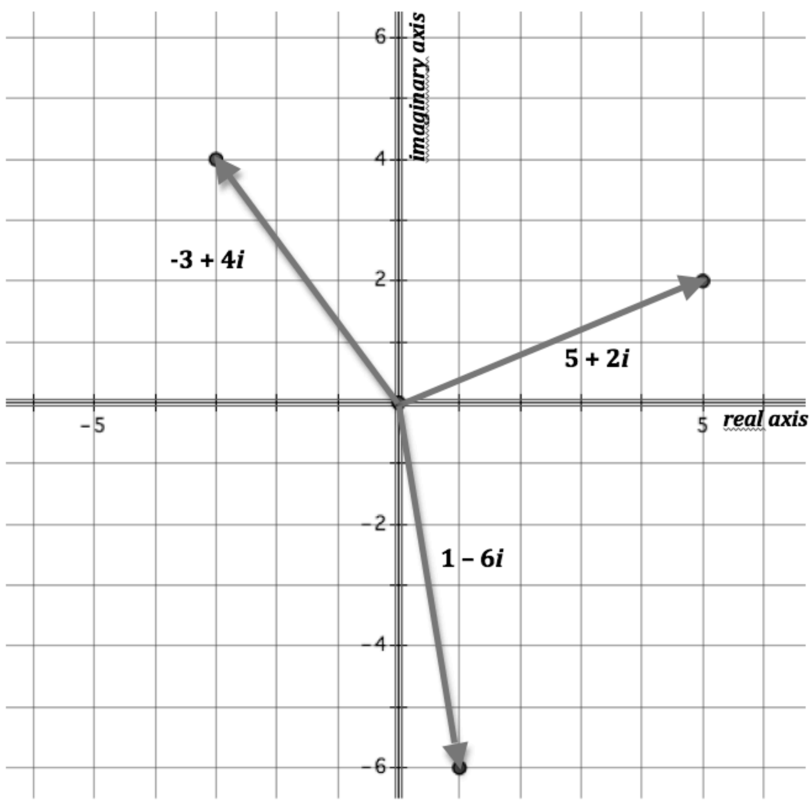
A Solidify Understanding Task



It is helpful to illustrate the arithmetic of complex numbers using a visual representation. To do so, we will introduce the complex plane.

As shown in the figure below, the complex plane consists of a horizontal axis representing the set of real numbers and a vertical axis representing the set of imaginary numbers. Since a complex number $a + bi$ has both a real component and an imaginary component, it can be represented as a point in the plane with coordinates (a, b). It can also be represented by a position vector with its tail located at the point (0, 0) and its head located at the point (a, b), as shown in the diagram. It will be useful to be able to move back and forth between both geometric representations of a complex number in the complex plane—sometimes representing the complex number as a single point, and sometimes as a vector.

You may want to review the Secondary Math 1 task, *The Arithmetic of Vectors*, so you can draw upon those ideas in the following work.



The Modulus of a Complex Number

It is often useful to be able to compare the magnitude of two different numbers. For example, collecting $25 in revenue will not pay off a $45 debt, since |25| < |-45|. Note that in this example we used the absolute value of signed numbers to compare the magnitude of the revenue and the debt. Since -45 lies farther from 0 along a real number line than 25, the debt is greater than the revenue. In a similar way, we can compare the relative magnitudes of complex numbers by determining how far they lie away from the origin, the point (0, 0), in the complex plane. We refer to the magnitude of a complex number as its **modulus**, and symbolize this length with the notation $|a + bi|$.

1. Find the modulus of each of the complex numbers shown in the figure above.

2. State a rule, either in words or using algebraic notation, for finding the modulus of any complex number $a + bi$.

Adding and subtracting complex numbers

3. Experiment with the vector representation of complex numbers to develop and justify an algebraic rule for adding two complex numbers: $(a + bi) + (c + di)$. How do your representations of addition of vectors on the complex plane help to explain your algebraic rule for adding complex numbers?

4. How would you represent the additive inverse of a complex number on the complex plane? How would you represent the additive inverse algebraically?

5. If we think of subtraction as adding the additive inverse of a number, use the vector representation of complex numbers to develop and justify an algebraic rule for subtracting two complex numbers: $(a + bi) - (c + di)$. How do your representations of the additive inverse of a complex number and the addition of vectors on the complex plane help to explain your algebraic rule for subtracting complex numbers?

Multiplying complex numbers

One way to think about multiplication on the complex plane is to treat the first factor in the multiplication as a scale factor.

6. Provide a few examples of multiplying a complex number by a real number scale factor: $a(c + di)$. For example, what happens to the vector representation of a complex number when the scale factor a is 4? ½? -2?

7. Provide a few examples of multiplying a complex number by an imaginary scale factor: $bi(c + di)$. For example, what happens to the vector representation when the scale factor bi is i? $2i$? $-3i$?

8. Experiment with the vector representation of complex numbers to justify the following rule for multiplying complex numbers:

$$(a + bi)(c + di) = a(c + di) + bi(c + di) = ac - bd + (ad + bc)i.$$

   How do the geometric observations you made in question 6, question 7 and question 3 show up in this work?

The conjugate of a complex number

The conjugate of a complex number $a + bi$ is the complex number $a - bi$. The conjugate of a complex number is represented with the notation $\overline{a + bi}$.

9. Illustrate an example of a complex number and its conjugate in the complex plane using vector representations.

10. Illustrate finding the sum of a complex number and its conjugate in the complex plane using vector representations.

11. Illustrate finding the product of a complex number and its conjugate in the complex plane using vector representations. (Use the geometric observations you made in questions 6-8 to guide your work.)

12. If z is a complex number and $\bar{z}$ is its conjugate, how are the moduli $|z|$ and $|\bar{z}|$ related?

13. Use either a geometric or algebraic argument to complete and justify the following statements for any complex number $a + bi$:

- The sum of a complex number and its conjugate is always the real number _________ .

- The product of a complex number and its conjugate is always the real number ________ .

The division of complex numbers

Dividing a complex number by a real number is the same as multiplying the complex number by the multiplicative inverse of the divisor. That is, $\frac{a+bi}{c} = \frac{1}{c}(a+bi) = \frac{a}{c} + \frac{b}{c}i$. Therefore, division of a complex number by a real number can be thought of in terms of multiplying the complex number by a real-valued scale factor, an idea we explored in question 6.

We have also observed that multiplying a complex number by its conjugate always gives us a real number result. We make use of this fact to change a problem involving division by a complex number into an equivalent problem in which the divisor is a real number.

14. Explain why $\frac{a+bi}{c+di}$ is equivalent to $\frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)}$.

15. Use this idea to find the quotient $\frac{3+5i}{4+2i}$.

We have been using a vector representation of complex numbers in the complex plane in the previous problems. In the following problems we will represent complex numbers simply as points in the complex plane.

Finding the distance between two complex numbers

To find the distance between two points on a real number line, we find the absolute value of the difference between their coordinates. (Illustrate this idea with a couple of examples.)

In a similar way, we define the distance between two complex numbers in the complex plane as the modulus of the difference between them.

16. Find the distance between the two complex numbers plotted on the complex plane below.

Finding the average of two complex numbers

The average of two real numbers $\frac{x_1 + x_2}{2}$ is located at the midpoint of the segment connecting the two real numbers on the real number line. (Illustrate this idea with a couple of examples.)

In a similar way, we define the average of two complex numbers to be the midpoint of the segment connecting the two complex numbers in the complex plane.

17. Find the average of the two complex numbers plotted on the complex plane below.