3.5 Throwing an Interception

A Develop Understanding Task

The *x*-intercept(s) of the graph of a function $f(x)$ are often very important because they are the solution to the equation $f(x) = 0$. In past tasks, we learned how to find the *x*-intercepts of the function by factoring, which works great for some functions, but not for others. In this task we are going to work on a process to find the *x*-intercepts of any quadratic function that has them. We'll start by thinking about what we already know about a few specific quadratic functions and then use what we know to generalize to all quadratic functions with *x*-intercepts.
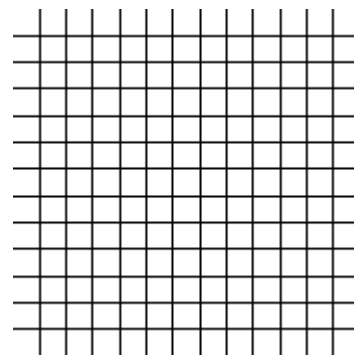
1. What can you say about the graph of the function $f(x) = x^2 - 2x - 3$?
    a. Graph the function

    b. What is the equation of the line of symmetry?

    c. What is the vertex of the function?

2. Now let's think specifically about the *x*-intercepts.
    a. What are the *x*-intercepts of $f(x) = x^2 - 2x - 3$?

   b. How far are the *x*-intercepts from the line of symmetry?

    c. If you knew the line of symmetry was the line *x = h*, and you know how far the *x*-intercepts are from the line of symmetry, how would you find the actual *x*-intercepts?

    d. How far above the vertex are the *x*-intercepts?

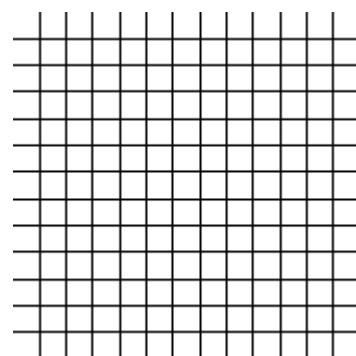   e. What is the value of $f(x)$ at the *x*-intercepts?

Just to make it a little easier to talk about some of the features that relate to the intercepts, let's name them with variables. From now on, when we talk about the distance from the line of symmetry to either of the x intercepts, we'll call it *d*. The diagram below shows this feature.
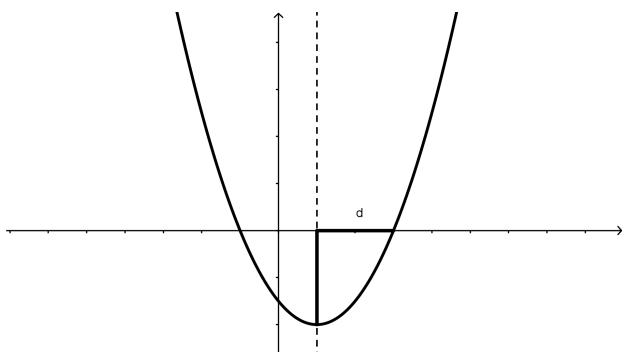
We will always refer to the line of symmetry as the line $x = h$, so the two x-intercepts will be at the points
$(h - d, 0)$ and $(h + d, 0)$.

3. So, let's think about another function: $f(x) = x^2 - 6x + 4$

   a. Graph the function by putting the equation into vertex form.

   b. What is the vertex of the function?

   c. What is the equation of the line of symmetry?

   d. What do you estimate the x-intercepts of the function to be?

   e. What do you estimate d to be?

   f. What is the value of $f(x)$ at the x-intercepts?

   g. Using the vertex form of the equation and your answer to "f" above, write an equation and solve it to find the exact values of the x intercepts.

   h. What is the exact value of d?

   i. Use a calculator to find approximations for the x-intercepts. How do they compare with your estimates?

4. What about a function with a vertical stretch? Can we find exact values for the x-intercepts the same way? Let's try it with: $f(x) = 2x^2 - 8x + 5$.

    a. Graph the function by putting the equation into vertex form.

    b. What is the vertex of the function?

    c. What is the equation of the line of symmetry?

    d. What do you estimate the x-intercepts of the function to be?

    e. What do you estimate d to be?

    f. What is the value of $f(x)$ at the x-intercepts?

    g. Using the vertex form of the equation and your answer to "f" above, write an equation and solve it to find the exact values of the x-intercepts.

    h. What is the exact value of d?

    i. Compare your solution to your estimate of the roots. How did you do?
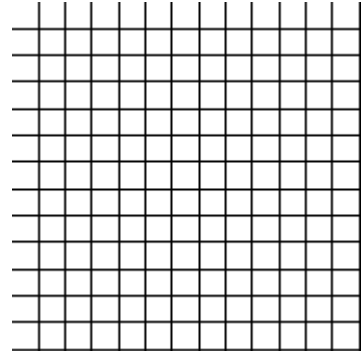
5. Finally, let's try to generalize this process by using: $f(x) = ax^2 + bx + c$ to represent any quadratic function that has x-intercepts. Here's a possible graph of $f(x)$.
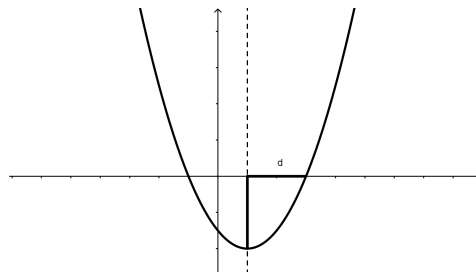
    a. Start the process the usual way by putting the equation into vertex form. It's a little tricky, but just do the same thing with a, b, and c as what you did in the last problem with the numbers.

b. What is the vertex of the parabola?

c. What is the line of symmetry of the parabola?

d. Write and solve the equation for the *x*-intercepts just as you did previously.

6. How could you use the solutions you just found to tell what the *x*-intercepts are for the function $f(x) = x^2 - 3x - 1$?

7. You may have found the algebra for writing the general quadratic function $f(x) = ax^2 + bx + c$ in vertex form a bit difficult. Here is another way we can work with the general quadratic function leading to the same results you should have arrived at in 5d.

   a. Since the two *x*-intercepts are *d* units from the line of symmetry *x = h*, if the quadratic crosses the *x*-axis its *x*-intercepts are at (*h* – *d*, 0) and (*h* + *d*, 0). We can always write the factored form of a quadratic if we know its *x*-intercepts. The factored form will look like $f(x) = a(x - p)(x - q)$ where *p* and *q* are the two *x*-intercepts. So, using this information, write the factored form of the general quadratic $f(x) = ax^2 + bx + c$ using the fact that its *x*-intercepts are at *h-d* and *h+d.*

   b. Multiply out the factored form (you will be multiplying two **trinomial** expressions together) to get the quadratic in standard form. Simplify your result as much as possible by combining like terms.

c. You now have the same general quadratic function written in standard form in two different ways, one where the **coefficients** of the terms are a, b and c and one where the coefficients of the terms are expressions involving a, h and d. Match up the coefficients; that is, b, the coefficient of x in one version of the standard form is equivalent to _____ in the other version of the standard form. Likewise c, the constant term in one version of the standard form is equivalent to _____ in the other.

d. Solve the equations b = _____ and c = _____ for h and d. Work with your equations until you can express h and d with expressions that only involve a, b and c.

e. Based on this work, how can you find the x-intercepts of any quadratic using only the values for a, b and c?

f. How does your answer to "e" compare to your result in 5d?

8. All of the functions that we have worked with on this task have had graphs that open upward. Would the formula work for parabolas that open downward? Tell why or why not using an example that you create using your own values for the coefficients a, b, and c.