3.8 To Be Determined . . .

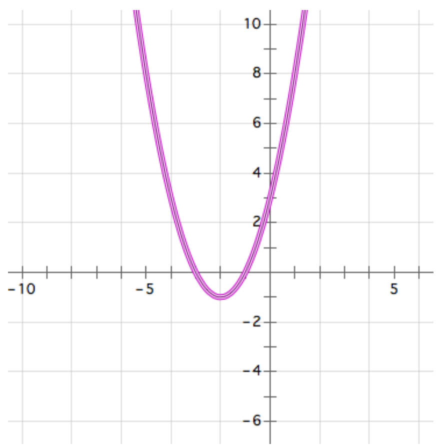
A Develop Understanding Task

Israel and Miriam are working together on a homework assignment. They need to write the equations of quadratic functions from the information given in a table or a graph. At first, this work seemed really easy. However, as they continued to work on the assignment, the algebra got more challenging and raised some interesting questions that they can't wait to ask their teacher.

Work through the following problems from Israel and Miriam's homework. Use the information in the table or the graph to write the equation of the quadratic function in all three forms. You may start with any form you choose, but you need to find all three equivalent forms. (If you get stuck, your teacher has some hints from Israel and Miriam that might help you.)

1.

x	y
-5	8
-4	3
-3	0
-2	-1
-1	0
0	3
1	8
2	15
3	24
4	35
5	48



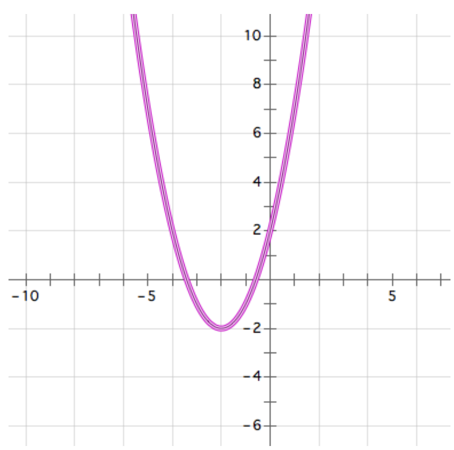
Standard form:

Factored form:

Vertex form:

2.

x	y
-5	7
-4	2
-3	-1
-2	-2
-1	-1
0	2
1	7
2	14
3	23
4	34
5	47



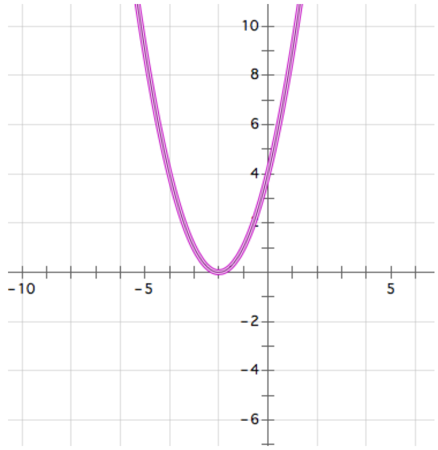
Standard form:

Factored form:

Vertex form:

3.

x	y
-5	9
-4	4
-3	1
-2	0
-1	1
0	4
1	9
2	16
3	25
4	36
5	49



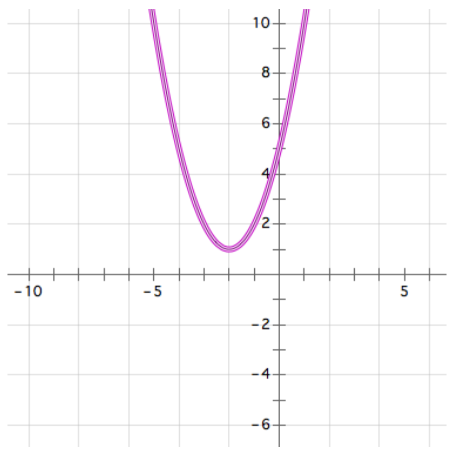
Standard form:

Factored form:

Vertex form:

4.

x	y
-5	10
-4	5
-3	2
-2	1
-1	2
0	5
1	10
2	17
3	26
4	37
5	50



Standard form:

Factored form:

Vertex form:

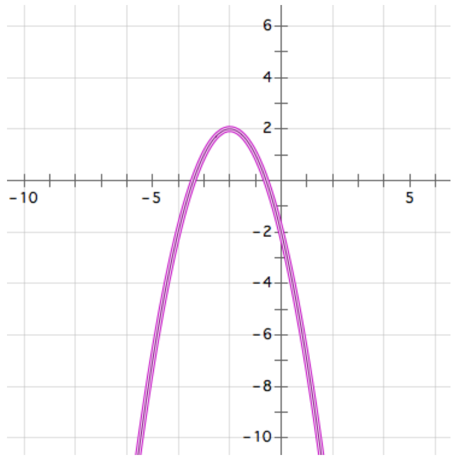
5. Israel was concerned that his factored form for the function in question 4 didn't look quite right. Miriam suggested that he test it out by plugging in some values for *x* to see if he gets the same points as those in the table. Test *<u>your</u>* factored form. Do you get the same values as those in the table?

6. Why might Israel be concerned about writing the factored form of the function in question 4?

Here are some more of Israel and Miriam's homework.

7.

x	y
-5	-7
-4	-2
-3	1
-2	2
-1	1
0	-2
1	-7
2	-14
3	-23
4	-34
5	-47



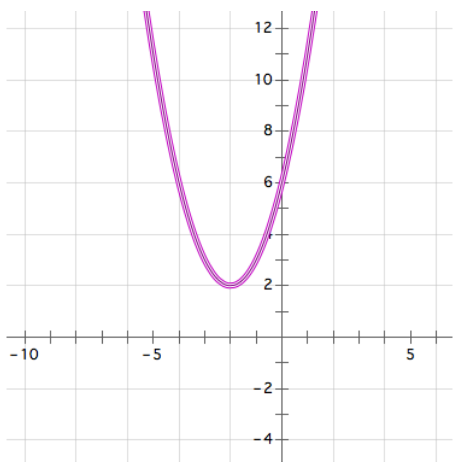
Standard form:

Factored form:

Vertex form:

8.

x	y
-5	11
-4	6
-3	3
-2	2
-1	3
0	6
1	11
2	18
3	27
4	38
5	51



Standard form:

Factored form:

Vertex form:

9. Miriam notices that the graphs of function 7 and function 8 have the same vertex point. Israel notices that the graphs of function 2 and function 7 are mirror images across the *x*-axis. What do you notice about the roots of these three quadratic functions?

The Fundamental Theorem of Algebra

A polynomial function is a function of the form:

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$

where all of the exponents are positive integers and all of the coefficients $a_0 \ldots a_n$ are constants.

As the theory of finding roots of polynomial functions evolved, a 17th century mathematician, Girard (1595-1632) made the following claim which has come to be known as the Fundamental Theorem of Algebra: *An nth degree polynomial function has n roots*.

In later math classes you will study polynomial functions that contain higher-ordered terms such as x^3 or x^5. Based on you work in this task, do you believe this theorem holds for quadratic functions? That is, do all functions of the form $y = ax^2 + bx + c$ always have two roots? [Examine the graphs of each of the quadratic functions you have written equations for in this task. Do they all have two roots? Why or why not?]

3.8 To Be Determined . . . – Teacher Notes

A Develop Understanding Task

Purpose: In the context of using procedures students have developed previously for writing equations for quadratic functions from the information given in a table or a graph, students will examine the nature of the roots of quadratic functions and surface the need for non-real roots when the quadratic function does not intersect the x-axis. This task follows the approach of the historical development of these non-real numbers. As mathematicians developed formulas for solving quadratic and cubic polynomials, the square root of a negative number would sometimes occur in their work. Although such expressions seemed problematic and undefined, when mathematicians persisted in working with these expressions using the same algebraic rules that applied to real-valued radical expressions, the work would lead to correct results. In this task, students will be able to write the equation of quadratic #4 in both vertex and standard form, but attempting to use the quadratic formula to find the roots, and therefore the factored form, will produce expressions that contain the square root of a negative number. However, if students persist in expanding out this factored form using the usual rules of arithmetic, the non-real-valued radical expressions will go away, leaving the same standard form as that obtained by expanding the vertex form. This should give some validity to these non-real-valued radical expressions. It is suggested that these numbers not be referred to as "imaginary" numbers in this task, but only that they are noted to be problematic in the sense of not representing a real value.

(Note: In the early history of mathematics even negative real numbers were considered "fictitious" or "false" solutions to quadratic and cubic equations, although Cardano (1501-1576) and Bombelli (1526-1572) also used square roots of negative numbers in their work. By the 17^{th} century negative numbers were recognized as legitimate solutions to polynomial equations, but complex numbers remained controversial through the 18^{th} century, even though they were useful in the theory of equations. Descartes (1596-1650) called all complex numbers "imaginary" and it was Euler (1707-1783) that introduced the symbol i for the square root of -1. Although expanding the number system to include complex numbers was sufficient to solve quadratic equations, it was not known if complex numbers were sufficient to solve cubic and higher-degree polynomial equations until 1799 when Gauss published a proof that all polynomial equations of degree n have n roots of the form $a + bi$. See a brief history of complex numbers at http://www.clarku.edu/~djoyce/complex/)

Core Standards Focus:

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N.CN.8 Extend polynomial identities to the complex numbers.

N.CN.9 Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A.REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b.

Note for Mathematics II A.REI.4a, A.REI.4b
Extend to solving any quadratic equation with real coefficients, including those with complex solutions.

Launch (Whole Class):

Explore (Small Group):
Watch for students who start question 1 by locating the x-intercepts of the graph (the zeroes in the table) and then using these values to write the two factors for the factored form. Once the factors are written they might multiply out the expression to get standard form, and then complete the square to get vertex form. This procedural approach works well for question 1, but will not work for questions 2 and 4, since the roots of the quadratic are not readily apparent from the table or the graph. For these students, suggest that Israel and Miriam noticed that all of the parabolas are symmetric about the same line $x = -2$, and so the vertex always lies on this line. Encourage these students to consider how they might use this fact to write the vertex form of each equation.

Watch for how students approach question 3, where the vertex lies on the x-axis. It may be difficult for students to recognize that the x-intercept at $x = -2$ is a root of multiplicity 2 and that the vertex form is $y=(x+2)^2$. For these students, suggest that Israel and Miriam noticed that as this sequence of parabolas are shifted up along the axis of symmetry the two x-intercepts get closer and closer together until they merge at $x = -2$. This would suggest that we have a “double root” at $x = -2$.

The most useful strategy students might use for question 2 is to write the vertex form by locating the minimum point in either the table or the graph. Once students have written the vertex form they can expand it to get standard form. They can then use the quadratic formula to find the zeros of the function, and then use these zeroes to write the corresponding factors. The irrational roots in question 2 can be found in this way. Have students verify that the radical values found using the quadratic formula fit in the intervals between -4 and -3 and between -1 and 0 by calculating approximate values for these roots using a calculator. In a similar way, students can find the roots of the quadratic in question 4 using the quadratic formula. At this point in time, allow students to write these roots as $-2-\sqrt{-1}$ and $-2+\sqrt{-1}$, you do not need to introduce the notation for complex numbers using i to represent the square root of -1. This will be the focus of the next task. Students may try to find approximate values for these roots using a calculator or recognize the dilemma of taking the square root of a negative number as being undefined in terms of real numbers. Acknowledge this dilemma, but also ask students to test their factored form for a few points (see question 5) and to multiply out their factored form in the usual way to verify that it yields the same standard form of the equation that they got when they expanded their vertex form. The goal here is to help students see that these numbers—while undefined in the real number system—yield correct results when manipulated with the familiar rules of algebra.

It is anticipated that students will get bogged down with this algebraic work. You can move to a whole class discussion during problems 4-6 to resolve these algebraic issues.

Discuss (Whole Class):
Begin the whole class discussion by examining question 4. First, have a student present the vertex form of this equation $y = (x+2)^2 + 1$. Then have a student present the standard form, which can be obtained by multiplying out the vertex form: $y = x^2 + 4x + 5$. Finally ask how we might write the factored form of this function, since it does not cross the *x*-axis and therefore has no *x*-intercepts. If you have identified students who used the quadratic formula to find the "zeros" or "roots" of the quadratic, have them present their work.

Students may have questions about how to write the roots of this quadratic after substituting values for *a, b* and *c* into the quadratic formula, or how to write the factored form since the roots contain two terms. Help support this algebraic work, including simplifying the radical expression $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1}$, and the rational expression $\frac{-4 \pm 2\sqrt{-1}}{2} = \frac{-4}{2} \pm \frac{2\sqrt{-1}}{2} = -2 \pm \sqrt{-1}$. The factored form is $y = \left[x - \left(-2 + \sqrt{-1}\right)\right] \cdot \left[x - \left(-2 - \sqrt{-1}\right)\right] = \left(x + 2 - \sqrt{-1}\right)\left(x + 2 + \sqrt{-1}\right)$. Once students have written the factored form, ask them to multiply out the two trinomial factors to obtain the standard form. Help them observe that since $\sqrt{-1} \cdot \sqrt{-1} = -1$ is consistent with the properties of radicals we have defined previously, and that this interpretation leads to the same standard form we started with. This consistency of properties will lead us in the next task to define the set of complex numbers.

Once the algebra of working with these negative radical expressions has been demonstrated, have students continue to work on the remainder of the task. Questions 7-9 point out that the algebraic work for irrational roots is similar to the algebraic work for these non-real roots.

Be sure to have a whole class discussion about the Fundamental Theorem of Algebra. Students may not feel like the theorem is true for quadratics, since #3 has only one real root and #4 and #8 have no real roots at all. Point out that we need to count these non-real roots, as well as multiple roots (such as $x = -2$ being a root of multiplicity 2 for question 3) to account for two roots for every quadratic. This will lead to the definition of complex numbers as roots in the next task.

Aligned Ready, Set, Go: Solving Quadratic and Other Equations 3.8