5.1 How Do You Know That?

A Develop Understanding Task

You probably know that the sum of the interior angles of any triangle is 180°. (If you didn't know that, you do now!) But an important question to ask yourself is, "How do you know that?"

We know a lot of things because we *accept it on authority*—we believe what other people tell us; things such as the distance from the earth to the sun is 93,020,000 miles or that the population of the United States is growing about 1% each year. Other things are just defined to be so, such as the fact that there are 5,280 feet in a mile. Some things we accept as true based on experience or repeated experiments, such as the sun always rises in the east, or "I get grounded every time I stay out after midnight." In mathematics we have more formal ways of deciding if something is true.

Experiment #1

1. Cut out several triangles of different sizes and shapes. Tear off the three corners (angles) of the triangle and arrange the vertices so they meet at a single point, with the edges of the angles (rays) touching each other like pieces of a puzzle. What does this experiment reveal about the sum of the interior angles of the triangles you cut out, and how does it do so?

2. Since you and your classmates have performed this experiment with several different triangles, does it guarantee that we will observe this same result for *all* triangles? Why or why not?

Experiment #2

Perhaps a different experiment will be more convincing. Cut out another triangle and trace it onto a piece of paper. It will be helpful to color-code each vertex angle of the original triangle with a different color. As new images of the triangle are produced during this experiment, color-code the corresponding angles with the same colors.

Locate the midpoints of each side of your cut out triangle by folding the vertices that form the endpoints of each side onto each other.

Rotate your triangle 180° about the midpoint of one of its sides. Trace the new triangle onto your paper and color-code the angles of this image triangle so that corresponding image/pre-image pairs of angles are the same color.

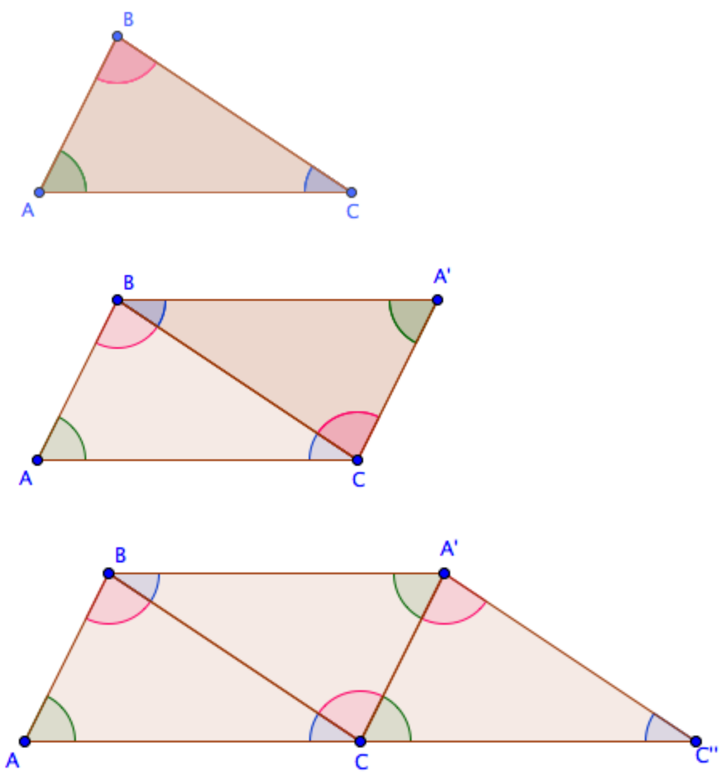
Now rotate the new "image" triangle 180° about the midpoint of one of the other two sides. Trace the new triangle onto your paper and color-code the angles of this new image triangle so that corresponding image/pre-image pairs of angles are the same color.

3. What does this experiment reveal about the sum of the interior angles of the triangles you cut out, and how does it do so?

4. Do you think you can rotate *all* triangles in the same way about the midpoints of its sides, and get the same results? Why or why not?

Examining the Diagram

Experiment #2 produced a sequence of triangles, as illustrated in the following diagram.



Here are some interesting questions we might ask about this diagram:

5. Will the second figure in the sequence always be a parallelogram? Why or why not?

6. Will the last figure in the sequence always be a trapezoid? Why or why not?

5.1 How Do You Know That – Teacher Notes
A Develop Understanding Task

Purpose: A major focus of the Mathematics II Geometry Standards is to develop the notion of formal proof—how the mathematics community comes to accept a statement as true. In this and subsequent tasks, students will explore a variety of ways of identifying the underlying reasoning behind a proof, and different formats for writing the proof. At the beginning of this sequence of tasks these proofs may take the form of informal verbal arguments. Over time, students should become more adept at constructing a logical sequence of statements that flow from beginning assumptions to justified conclusions.

In this task students consider the question, "How do you know something is true?" In mathematics, two different types of reasoning are used: inductive reason is the process of examining many examples, noticing a pattern, and stating a conjecture. Deductive reasoning is the process of starting with statements assumed or accepted as true (generally from a previous sequence of deductive reasoning), then creating a logical sequence of statements (if a is true, then b is true; if b is true, then c is true; if c is true, then d is true, etc.), until we arrive at the desired conclusion. Hence, inductive reasoning surfaces conjectures that need to be verified by deductive reasoning. Much of what is proved to be true by deductive reasoning in Mathematics II has already surfaced through experimentation in previous courses. This task should help students distinguish between accepting something as true based on experience or experimentation, and knowing something is true based on logical reasoning.

In this task students explore two different ways of knowing that the sum of the angles in a triangle is 180°, one based on experiments with specific triangles, and one based on a transformational argument that can be applied to all triangles.

Core Standards Focus:

G.CO.10 Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°.

Mathematics II Note for G.CO.10: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

Related Standards: G.CO.11

Launch (Whole Class): [Experiment 1]
Ask students if they know what sum they will get if they add up the three angles in any triangle. It is likely that most will know that the sum is 180°. Ask them how they know that is true. Is it

something they can recall from memory, but can't explain? Do they justify their argument by referring to a specific triangle, such as an equilateral triangle? How are they convinced that this property is true for *all* triangles? Perhaps they will refer to some prior experience in a previous math class, such as experiment #1 described in the task.

Point out that we have multiple "ways of knowing." (adapted from Schifter, D., Bastable, V., & Russell, S.J.. *Developing Mathematical Ideas: Reasoning Algebraically About Operations Casebook*, chapter 8). List these ways on the board for reference during this task:

- Accepting it on authority
- Trying it out with multiple examples
- Making an argument based on a diagram or representation
- Making an argument based on previously proved statements and logical ways of reasoning

Point out that most know that the sum of the angles of a triangle is 180° based on having been told by a previous teacher, or perhaps they have had an opportunity to try it out with examples. Describe experiment #1 and let students carry out the experiment, including responding to the two questions. (If students have already had this experience previously, model it with a example or two, and move to the discussion of what we can "know" from this experiment.)

Explore (Small Group): [Experiment 1]
Observe that students are carrying out the experiment correctly: rearranging the vertices torn from the corners of the triangle to form a line. Students should be able to describe that because the three angles come together to form a straight line that the angles sum to 180°.

Discuss (Whole Class): [Experiment 1]
Have a short whole class discussion about what this experiment "proves." Helps students recognize that each individual experiment illustrates that the claim holds for that specific triangle, and even though we have tried several different triangles we haven't tried them all. How do we know that we haven't found one of the triangles for which this statement may not be true? Maybe it isn't true for really large triangles, or perhaps really small triangles, or some really "odd" looking triangles. We really don't know, since we haven't tried them all.

Launch (Whole Class): [Experiment 2]
Introduce experiment #2. An online app is available at http://geogebracentral.blogspot.com/2011/01/triangle-angle-sum-proof.html that can be used to demonstrate the rotation about the midpoint of one of the sides of the triangle. If available, show this app to model the activity, then give students a triangle, a white sheet of paper on which to trace their rotated triangles, and colored pencils to keep track of the three different vertices of their triangles.

Explore (Small Group): [Experiment 2]
Students should color each vertex of their triangle with a different color, and then trace their triangle onto the white paper close to one edge of the paper—so they have room on the paper to

draw two additional rotations of the triangle. Watch that they color-code each corresponding vertex of their triangles with the same color. Make sure that they have accurately identified the midpoints of the sides of their triangle, and are rotating their triangle about a midpoint—not reflecting their triangles over a side.

All students should answer questions 3 and 4 that accompany the experiment, and you should allow time for at least some students to work on the "Examining the Diagram" questions.

Discuss (Whole Class): [Experiment 2]
Focus the discussion on question 4, leading to an argument that is based on the rotation of the triangles, and therefore independent of the specific starting triangle. The sequence of images in the "Examining the Diagram" section of the task lay out the general argument: rotating a triangle about the midpoint of one its sides forms a parallelogram; rotating this image a second time forms a trapezoid in which we can see a translated image of the original triangle, with one side of the original triangle extended to form a straight line. This straight line consists of images of the three angles of the triangle, verifying that the sum of the angles of the original triangle is 180°.

There are some subtle issues with this general argument, as pointed out in questions 5 and 6. To know that a straight line exists through points *A*, *C*, and *C''* we need to know that figure *ABA'C* and figure *CBA'C''* are parallelograms that share a common side—therefore, $\overline{AC}$ and $\overline{CC'}$ are either parallel to each other or on the same line—and since the altitudes of both parallelograms are the same length (the altitude of the original triangle), the segments are the same distance from $\overline{BA'}$ and are therefore on the same line. Students might be able to argue that the two quadrilaterals are parallelograms based on the property that opposite sides are congruent. This is a property that they may have observed in Mathematics I, when they experimented with symmetries of quadrilaterals. This is a property that will be proved later in this module. Therefore, at this point, our way of knowing may be more at the level of "making an argument based on a diagram or representation" rather than "making an argument based on previously proved statements."

Aligned Ready, Set, Go: Geometric Figures 5.1